

# Equivalence relations for the classical capacity of single-mode Gaussian quantum channels

Joachim Schäfer,<sup>1</sup> Evgueni Karpov,<sup>1</sup> Raúl García-Patrón,<sup>1,2</sup> Oleg V. Pilyavets,<sup>1</sup> and Nicolas J. Cerf<sup>1</sup>

<sup>1</sup>QuIC, Ecole Polytechnique de Bruxelles, CP 165, Université Libre de Bruxelles (ULB), 1050 Brussels, Belgium

<sup>2</sup>Max-Planck-Institut für Quantenoptik, Hans-Kopfermann-Straße 1, 85748 Garching, Germany

(Dated: March 21, 2013)

We prove the equivalence of an arbitrary single-mode Gaussian quantum channel and a newly defined *fiducial* channel preceded by a phase shift and followed by a Gaussian unitary operation. This equivalence implies that the classical capacity and Gaussian classical capacity of any single-mode Gaussian channel can be calculated based on this fiducial channel, which is furthermore simply realizable with a beam-splitter, two identical single-mode squeezers, and a two-mode squeezer. In a large domain of parameters, we also provide an analytical expression for the Gaussian classical capacity, exploiting its additivity, and prove that the classical capacity does not exceed it by more than  $1/\ln 2$  bits.

**Introduction.**—Quantum channels play a key role in quantum information theory. In particular, bosonic Gaussian channels model most physical communication links, such as optical fibers or free space information transmission [1, 2]. One of the central characteristics of quantum channels is their classical capacity, i.e. the maximal amount of classical bits that can be transmitted per use of the channel. A lot of attention has already been devoted to the study of the classical capacity of Gaussian channels [3–23]. Since Gaussian encodings are more relevant for experimental implementations, easier to work with analytically, and conjectured to be optimal [16], the so-called Gaussian classical capacity was introduced and evaluated for specific Gaussian channels [16–23].

In this Letter, we greatly simplify the calculation of these capacities [42] for an arbitrary single-mode Gaussian channel. Namely, we show that any single-mode Gaussian channel is indistinguishable from a newly defined *fiducial* channel, preceded by a phase-shift and followed by a general Gaussian unitary. Since neither the phase-shift at the channel’s input nor the Gaussian unitary at the channel’s output change the output entropy or affect the input energy constraint, we conclude that the capacities of this channel are equal to those of the fiducial channel, which furthermore, is completely specified by three parameters. This conclusion also holds for any cascade of Gaussian channels since the latter is equivalent to another Gaussian channel. Our results allow us to go beyond previous works on the Gaussian capacity [20–22] and provide its analytical expression for any Gaussian channel in some energy range, where it is additive. Then, we prove that the classical capacity does not exceed the Gaussian capacity by more than  $1/\ln 2$  bits (generalizing [24]), which furthermore becomes the actual capacity if the minimum output entropy conjecture for phase-insensitive Gaussian channels [16, 25] is true.

**Gaussian channel.**—Let  $\hat{\rho}^G(\alpha, V)$  be an  $n$ -mode Gaussian state, where the coherent vector  $\alpha \in \mathbb{R}^{2n}$  and the  $2n \times 2n$  covariance matrix (CM)  $V$  are the first- and second-order moments of the  $2n$  dimensionless quadratures, respectively, with  $\hbar = 1$ . Then, a Gaussian channel  $\Phi$  is a completely-positive trace-preserving map which is closed on the set of Gaussian states [16]. It transforms Gaussian states as  $\Phi[\hat{\rho}^G(\alpha_{\text{in}}, V_{\text{in}})] = \hat{\rho}^G(\alpha_{\text{out}}, V_{\text{out}})$ , with the input mo-

ments  $\{\alpha_{\text{in}}, V_{\text{in}}\}$  being connected to the output moments  $\{\alpha_{\text{out}}, V_{\text{out}}\}$  according to

$$\alpha_{\text{out}} = X\alpha_{\text{in}} + \delta, \quad V_{\text{out}} = XV_{\text{in}}X^T + Y, \quad (1)$$

where  $\delta$  is the displacement introduced by the channel,  $X$  is a  $2n \times 2n$  real matrix, and  $Y$  is a  $2n \times 2n$  real, symmetric, and non-negative matrix. For simplicity, we choose  $\delta = 0$  in what follows (the capacity is not affected by first-order moments), and focus on the action of the map  $\Phi$  on second-order moments using the simplified notation  $\Phi(V_{\text{in}}) = V_{\text{out}}$ . Then, the map  $\Phi$  is fully characterized by matrices  $X$  and  $Y$ , which must satisfy the relation  $Y + \frac{i}{2}(\Omega - X\Omega X^T) \geq 0$  [16], where

$$\Omega = \bigoplus_{k=1}^n \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (2)$$

is the symplectic form [26]. In the following, we focus on single-mode channels ( $n = 1$ ) and use the parameters

$$\tau = \det X, \quad y = \sqrt{\det Y}, \quad (3)$$

where  $\tau$  may be a channel transmissivity (if  $\tau \leq 1$ ) or amplification gain (if  $\tau \geq 1$ ), while  $y$  reflects the added noise. The map  $\Phi$  describes a quantum channel if  $y \geq \frac{1}{2}|\tau - 1|$  [27], and it is an *entanglement breaking* channel if  $y \geq \frac{1}{2}(|\tau| + 1)$  [28]. The single-mode Gaussian channels can therefore be conveniently represented in a  $(\tau, y)$  plane, see Fig. 1.

**Canonical decomposition.**—Any single-mode Gaussian channel  $\Phi$  can be decomposed as  $\Phi = U_2 \circ \Phi^C \circ U_1$ , where  $U_1$  and  $U_2$  are Gaussian unitaries, and  $\Phi^C$  is a *canonical* channel characterized by the matrices  $(X_C, Y_C)$  [29–31]. The action of a Gaussian unitary  $U$  on a Gaussian state can be completely specified by a symplectic transformation  $M$  acting on the second-order moments of the state (we ignore first-order moments), so that the canonical decomposition may be written as  $(U_2 \circ \Phi^C \circ U_1)(V_{\text{in}}) = M_2 \Phi^C(M_1 V_{\text{in}} M_1^T) M_2^T$ . It is known that there are seven classes of canonical channels  $\Phi^C$  (see Table I) [29–31]. We first discuss these canonical channels  $\Phi^C$  and give explicit relations for the canonical decomposition of an arbitrary Gaussian channel  $\Phi$  (see also the supplemental material [43]).

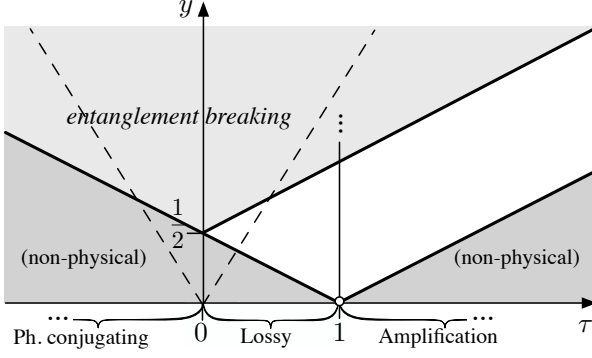


FIG. 1. Admissible regions in the parameter space  $(\tau, y)$  for Gaussian quantum channels. Each thermal channel  $\Phi_{(\tau,y)}^{\text{TH}}$  is associated with a point. The vertical line at  $\tau = 0$  corresponds to the zero-transmission channel as well as the classical signal channel  $\Phi^{\text{CS}}$ . The vertical line at  $\tau = 1$  corresponds to the classical additive-noise channel. The limiting case  $(\tau = 1, y = 0)$  is the perfect transmission channel as well as the single quadrature classical noise channel  $\Phi^{\text{SQ}}$ . The Gaussian capacity of  $\Phi_{(\tau,y,s)}^{\text{F}}$  is additive if  $\bar{N} \geq \bar{N}_{\text{thr}}$ . This is equivalent to  $y \leq y_{\text{thr}} = |\tau|(e^{-2s}(1 + 2\bar{N}) - 1)/(1 - e^{-4s})$ . An example of  $y_{\text{thr}}$  is given by the dashed line, for particular values of  $\bar{N}$  and  $s$ .

The first five channels in Table I can be treated together, and we refer to them collectively as *thermal* channels, denoted by  $\Phi_{(\tau,y)}^{\text{TH}}$ . They are defined as

$$\mathbf{X}_{\text{TH}} = \begin{pmatrix} \sqrt{|\tau|} & 0 \\ 0 & \text{sgn}(\tau)\sqrt{|\tau|} \end{pmatrix}, \quad \mathbf{Y}_{\text{TH}} = \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix}, \quad (4)$$

where  $\text{sgn}(\tau) = -1$  if  $\tau < 0$  and  $\text{sgn}(\tau) = 1$  if  $\tau \geq 0$ . As shown in Fig. 2 (a), the channel  $\Phi^{\text{TH}}$  can be physically realized by a beamsplitter with transmissivity  $T$  followed by a phase-insensitive two-mode squeezer (TMS) with gain  $G$  [27]. For the *zero-transmission* ( $\tau = 0$ ), *lossy* ( $0 \leq \tau \leq 1$ ), *amplification* ( $\tau \geq 1$ ), and *classical additive-noise channel* ( $\tau = 1$ ), the output is given by the signal's output of the TMS, and these four canonical channels in Table I correspond to *phase-insensitive* channels. For the fifth canonical channel, i.e., the *phase-conjugating channel* ( $\tau < 0$ ), the output is given by the idler's output of the TMS. These five channels share the property that they map any thermal state to a thermal state, therefore, we call them thermal channels. Each particular channel  $\Phi^{\text{TH}}$  with parameters  $(\tau, y)$  corresponds to a single point in Fig. 1, while the relations between  $(\tau, y)$  and  $(T, G)$  are given in Table I.

The sixth canonical channel is the *classical signal* (or quadrature erasing) channel, which we denote by  $\Phi^{\text{CS}}$ . Its action is defined by

$$\mathbf{X}_{\text{CS}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{Y}_{\text{CS}} = \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix}, \quad y \geq \frac{1}{2}. \quad (5)$$

This channel can be physically implemented with a continuous-variable controlled-NOT (C-NOT) gate [32, 33]. The corresponding scheme is depicted in Fig. 2 (b), where

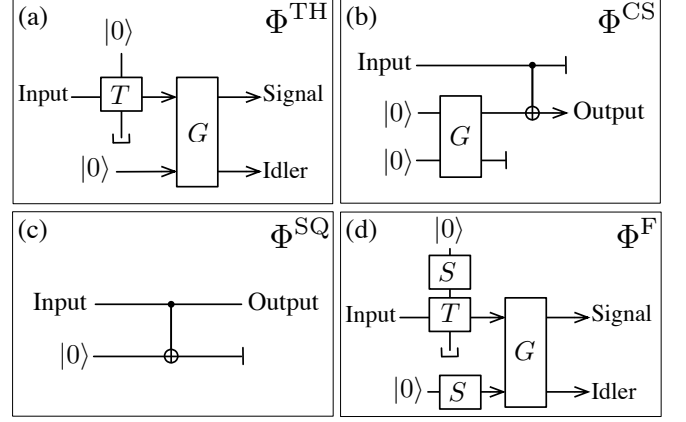


FIG. 2. Physical realization of (a) the thermal channel  $\Phi^{\text{TH}}$ , (b) the classical signal channel  $\Phi^{\text{CS}}$ , (c) the single-quadrature classical noise channel  $\Phi^{\text{SQ}}$ , and (d) the fiducial channel  $\Phi^{\text{F}}$ .  $T$  stands for a beamsplitter with transmissivity  $T$ ,  $G$  for a two-mode squeezer with gain  $G$ ,  $S$  for a single-mode squeezer,  $|0\rangle$  for the vacuum state, and “ $\dashv$ ” denotes “tracing out” the mode.

$G = y + 1/2$ . Note that  $\tau = 0$ , implying that  $\Phi^{\text{CS}}$  is always entanglement breaking (see Fig. 1). The seventh and last canonical channel is the *single-quadrature classical noise channel*, which we name  $\Phi^{\text{SQ}}$ . Its action is described by the matrices

$$\mathbf{X}_{\text{SQ}} = \mathbb{I}, \quad \mathbf{Y}_{\text{SQ}} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad (6)$$

which corresponds to  $\tau = 1$  and  $y = 0$  (as the perfect transmission channel). It is known that  $\Phi^{\text{SQ}}$  is not entanglement breaking [28]. It is shown in supplemental material [43] that channels  $\Phi^{\text{CS}}$  and  $\Phi^{\text{SQ}}$  can be represented by schemes Fig. 2 (b) and (c), respectively.

*Fiducial channel.*— Now, our central point is that the above canonical decomposition is not always useful for evaluating capacities of bosonic channels with input energy constraint (which is needed, otherwise the capacities are infinite). Indeed, the Gaussian unitary  $U_1$  that precedes the canonical channel  $\Phi^{\text{C}}$  changes, in general, the input energy. Therefore, we introduce a new decomposition in terms of a *fiducial channel*  $\Phi^{\text{F}}$ , where the preceding unitary is passive and does not affect the input energy restriction. Below we show that this decomposition has the major advantage that the energy-restricted capacity of any Gaussian channel reduces to that of the fiducial channel  $\Phi^{\text{F}}$ . The latter generalizes  $\Phi^{\text{TH}}$  by adding a squeezing to the effective added noise, that is

$$\mathbf{X}_{\text{F}} = \mathbf{X}_{\text{TH}}, \quad \mathbf{Y}_{\text{F}} = y \begin{pmatrix} e^{2s} & 0 \\ 0 & e^{-2s} \end{pmatrix}. \quad (7)$$

Thus, it depends on three parameters  $(\tau, y, s)$ , and will be denoted by  $\Phi_{(\tau,y,s)}^{\text{F}}$ . This channel can be physically realized by the setup depicted in Fig. 2 (d), where the “idler” corresponds again to the output of the phase-conjugating channel

Channel	Symbol	Class	$\mathbf{X}_C$	$\mathbf{Y}_C$	$\tau$	Domain of $\tau$	Domain of $y$
Zero-Transmission	$\mathcal{A}_1$	$\Phi^{\text{TH}}$	0	$(G - 1/2)\mathbb{I}$	0	0	$[1/2, \infty)$
Classical additive noise	$\mathcal{B}_2$		$\mathbb{I}$	$(G - 1)\mathbb{I}$	$TG = 1$	1	$[0, \infty)$
Lossy	$\mathcal{C}_L$		$\sqrt{\tau}\mathbb{I}$	$[G(1 - T/2) - 1/2]\mathbb{I}$	$TG$	$[0, 1]$	$[(1 - \tau)/2, \infty)$
Amplification	$\mathcal{C}_A$		$\sqrt{\tau}\mathbb{I}$	$[G(1 - T/2) - 1/2]\mathbb{I}$	$TG$	$[1, \infty)$	$[(\tau - 1)/2, \infty)$
Phase conjugating	$\mathcal{D}$		$\sqrt{ \tau }\sigma_z$	$[(1 - T)(G - 1) + G]/2\mathbb{I}$	$-T(G - 1)$	$(-\infty, 0]$	$[(1 - \tau)/2, \infty)$
Classical-signal	$\mathcal{A}_2$	$\Phi^{\text{CS}}$	$(\mathbb{I} + \sigma_z)/2$	$(G - 1/2)\mathbb{I}$	0	0	$[1/2, \infty)$
Single-quad. cl. noise	$\mathcal{B}_1$	$\Phi^{\text{SQ}}$	$\mathbb{I}$	$(\mathbb{I} - \sigma_z)/4$	1	1	0

TABLE I. Canonical channels  $\Phi^C$  with corresponding symbols as defined in [29–31], new representation in terms of  $\Phi^{\text{TH}}$ ,  $\Phi^{\text{CS}}$ ,  $\Phi^{\text{SQ}}$  and the corresponding matrices  $(\mathbf{X}_C, \mathbf{Y}_C)$ , where  $\sigma_z = \text{diag}(1, -1)$ . The parameter  $T \in [0, 1]$  is the transmissivity of the beamsplitter and the parameter  $G \geq 1$  is the gain of the two-mode squeezer shown in the physical schemes in Fig. 2.

and the “signal” to that of the other channels. In the case of  $0 \leq \tau \leq 1$ , this channel corresponds to the mixing of the input state with an arbitrary squeezed thermal state on a beam-splitter with transmissivity  $\tau$ . The fiducial channel  $\Phi^F$  can be used to decompose any Gaussian channel  $\Phi$  (at least in a proper limit):

**Theorem 1.** *For a single-mode Gaussian channel  $\Phi$  defined by matrices  $\mathbf{X}$  and  $\mathbf{Y}$  with  $\tau \neq 0$  and  $y > 0$ , there exists a fiducial channel  $\Phi^F$  defined by matrices  $\mathbf{X}_F(\tau)$ ,  $\mathbf{Y}_F(y, s)$  with  $\tau$  and  $y$  obtained from Eq. (3), a symplectic transformation  $\mathbf{M}$ , and a rotation in phase space  $\Theta$  such that*

$$\mathbf{X} = \mathbf{M} \mathbf{X}_F(\tau) \Theta, \mathbf{Y} = \mathbf{M} \mathbf{Y}_F(y, s) \mathbf{M}^\top, \quad (8)$$

where the explicit dependencies of  $\mathbf{M}$ ,  $\Theta$ , and  $s$  on the parameters of the channel  $\Phi$  are presented in the supplemental material in Eqs.(26)-(36).

*Proof.* We only sketch the proof here (see [43] for the full proof). First, one finds matrices  $\Theta_Y$  and  $S_Y$  such that  $S_Y^{-1} \Theta_Y^\top \mathbf{Y} \Theta_Y S_Y^{-1} = \text{diag}(y, y)$ , where  $\Theta$  and  $S$  denote matrices corresponding to a rotation and a symplectic squeezing operation, respectively. Secondly, one obtains the singular value decomposition  $\mathbf{X} = \Theta_{1X} S_X \mathbf{X}_F \Theta_{2X}$ , where  $\mathbf{X}_F$  reads as in Eq. (7). Then one defines  $\mathbf{M} = \Theta_{1X} S_X \Theta_F^\top$ , where  $\Theta_F$  is found such that  $\mathbf{M}^{-1} \mathbf{Y} \mathbf{M} = \mathbf{Y}_F = y \text{diag}(e^{2s}, e^{-2s})$ . The squeezing parameter  $s$  depends on all angles and squeezing operations  $S_X, S_Y$ . Finally, one introduces  $\Theta_F$  in  $\mathbf{X}$ , i.e.  $\mathbf{X} = \Theta_{1X} S_X \Theta_F^\top \Theta_F \mathbf{X}_F \Theta_{2X} = \mathbf{M} \mathbf{X}_F \Theta$ , where  $\Theta$  depends on  $\Theta_{1X}, \Theta_F$  and the sign of  $\tau$ . Despite Theorem 1 requires that  $\tau \neq 0$  and  $y > 0$ , or, equivalently, that  $\text{rank}(\mathbf{X}) = \text{rank}(\mathbf{Y}) = 2$ , it can be extended to lower-rank cases with minor modifications as shown in [43].  $\square$

**Capacities.**—We now focus on the capacity  $C$  of the Gaussian channel  $\Phi$  with a constrained mean photon number  $\bar{N}$  at the input. It is defined as the maximal amount of bits that can be transmitted per use of the channel  $\Phi$ , i.e. [15, 34]

$$C(\Phi, \bar{N}) = \lim_{n \rightarrow \infty} \frac{1}{n} C_\chi(\Phi^{\otimes n}, n\bar{N}), \quad (9)$$

where  $n$  is the number of channel uses, and  $C_\chi(\Phi, \bar{N})$  is the *one-shot* capacity of the channel, i.e.

$$C_\chi(\Phi, \bar{N}) = \max_{\mu: \hat{\rho} \in \mathcal{E}_{\bar{N}}} \chi(\Phi, \mu), \quad (10)$$

$$\chi(\Phi, \mu) = S(\Phi[\hat{\rho}]) - \int \mu(dx) S(\Phi[\hat{\rho}_x]).$$

Here  $S(\hat{\rho}) = -\text{Tr}(\hat{\rho} \log_2 \hat{\rho})$  is the von Neumann entropy. The maximum is taken over all probability measures  $\mu(x)$  in the whole space  $\mathcal{H}$  of pure states  $\hat{\rho}_x$  of the field mode and  $\hat{\rho} = \int \mu(dx) \hat{\rho}_x$  belongs to the set  $\mathcal{E}_{\bar{N}}$  of states which have a mean photon number not greater than  $\bar{N}$ . Since it was shown that, in general, the one-shot capacity is not additive [35], one has to take the limit in (9), unless additivity is explicitly proven for the given channel. The decomposition stated in Theorem 1 implies:

**Corollary 1.** *For a single-mode Gaussian channel  $\Phi$  with parameters  $(\tau \neq 0, y > 0)$ , there exists a fiducial channel  $\Phi^F$  as defined in Theorem 1 such that*

$$C(\Phi, \bar{N}) = C(\Phi^F, \bar{N}). \quad (11)$$

*Proof.* First, the symplectic transformation  $\mathbf{M}$  that follows  $\Phi^F$  in Theorem 1 does not change the entropies in the definition of  $\chi$ . Second, there is no energy constraint on the output of the channel. Hence,  $\mathbf{M}$  can be dropped. Then, the rotation  $\Theta$  preceding  $\Phi^F$  in Theorem 1 may be regarded as change of a reference phase that can be chosen arbitrarily; therefore,  $\Theta$  can be dropped as well. Thus,  $C_\chi(\Phi, \bar{N}) = C_\chi(\Phi^F, \bar{N})$  holds. To evaluate the one-shot capacity of  $n$  copies of  $\Phi$  the same reasoning can be applied, where now the preceding and following transformations are given by  $\oplus_{i=1}^n \mathbf{M}$  and  $\oplus_{i=1}^n \Theta$ , respectively. Hence, it follows that  $C_\chi(\Phi^{\otimes n}, n\bar{N}) = C_\chi((\Phi^F)^{\otimes n}, n\bar{N})$  which together with Eq. (9) implies Eq. (11). Note that despite Eq. (11) requires  $\tau \neq 0$  and  $y > 0$ , it can be easily extended to the general case [43].  $\square$

Corollary 1 has also other implications. If the corresponding fiducial channel  $\Phi_{(\tau, y, s)}^F$  is entanglement breaking, i.e.  $y \geq (|\tau| + 1)/2$ , then its one-shot capacities of both  $\Phi_{(\tau, y, s)}^F$  and  $\Phi$  are additive [36, 37], namely  $C(\Phi, \bar{N}) = C_\chi(\Phi^F, \bar{N})$ .

*Gaussian capacities.*—For experimental implementations and analytical calculations, it is useful to focus on Gaussian symbols and encodings. We call the capacity restricted to Gaussian encodings the Gaussian capacity [20–22]  $C^G$  and define it as

$$\begin{aligned} C^G(\Phi, \bar{N}) &= \lim_{n \rightarrow \infty} \frac{1}{n} C_\chi^G(\Phi^{\otimes n}, n\bar{N}), \\ C_\chi^G(\Phi, \bar{N}) &= \max_{\mu^G: \hat{\rho}^G \in \mathcal{E}_N^G} \chi(\Phi, \mu^G), \end{aligned} \quad (12)$$

where  $C_\chi^G(\Phi, \bar{N})$  is the *one-shot Gaussian capacity*. The maximum is taken now over all measures  $\mu^G(\alpha, \mathbf{V})$  for which  $\hat{\rho}^G(\bar{\alpha}_{\text{in}}, \bar{\mathbf{V}}_{\text{in}}) = \int \mu^G(d\alpha, d\mathbf{V}) \hat{\rho}^G(\alpha, \mathbf{V})$  is in the set  $\mathcal{E}_N^G$  of Gaussian states with a mean photon number not greater than  $\bar{N}$ . Unlike previous works (e.g., [17]), we require the individual symbol states as well as the averaged state to be Gaussian. Then we prove that the one-shot Gaussian capacity of an arbitrary single-mode Gaussian channel  $\Phi$  is given by the well known expression [4]:

$$\begin{aligned} C_\chi^G(\Phi, \bar{N}) &= \max_{\mathbf{V}_{\text{in}}, \mathbf{V}_{\text{mod}}} \{ \chi^G(\bar{\nu}, \nu) \mid \text{Tr}[\mathbf{V}_{\text{in}} + \mathbf{V}_{\text{mod}}] \leq 2\bar{N} + 1 \}, \\ \chi^G(\bar{\nu}, \nu) &= g\left(\bar{\nu} - \frac{1}{2}\right) - g\left(\nu - \frac{1}{2}\right), \\ g(x) &= (x+1) \log_2(x+1) - x \log_2 x, \end{aligned} \quad (13)$$

where  $\mathbf{V}_{\text{in}}$  is the CM of a pure Gaussian input state  $\hat{\rho}^G(0, \mathbf{V}_{\text{in}})$  fulfilling  $\det(2\mathbf{V}_{\text{in}}) = 1$ ,  $\mathbf{V}_{\text{mod}}$  is the CM of a classical Gaussian distribution used to displace the input state and to generate the modulated input state  $\hat{\rho}^G(0, \bar{\mathbf{V}}_{\text{in}})$  whose CM is  $\bar{\mathbf{V}}_{\text{in}} = \mathbf{V}_{\text{in}} + \mathbf{V}_{\text{mod}}$  with  $\text{Tr}[\bar{\mathbf{V}}_{\text{in}}] \leq 2\bar{N} + 1$ . Furthermore,  $\nu = \sqrt{\det \mathbf{V}_{\text{out}}}$  and  $\bar{\nu} = \sqrt{\det \bar{\mathbf{V}}_{\text{out}}}$  are the symplectic eigenvalues of the output and modulated output states with CM  $\mathbf{V}_{\text{out}} = \Phi(\mathbf{V}_{\text{in}})$  and  $\bar{\mathbf{V}}_{\text{out}} = \Phi(\bar{\mathbf{V}}_{\text{in}})$ , respectively. The detailed derivation is provided in [43].

The one-shot Gaussian capacity is equal to the Gaussian capacity, i.e.  $C^G(\Phi, \bar{N}) = C_\chi^G(\Phi, \bar{N})$ , if it is additive. Interestingly, such an additivity can be proven if the input energy exceeds some threshold  $\bar{N}_{\text{thr}}$  (see [43]). Note that [17] also derives additivity but for a slightly different definition of  $C_\chi^G$  and without respecting the energy threshold. In addition, an analog of Corollary 1 can easily be shown to hold for Gaussian capacities, namely  $C^G(\Phi, \bar{N}) = C^G(\Phi^F, \bar{N})$ . Therefore, using the fiducial channel  $\Phi^F$ , we can analytically find the Gaussian capacity of any Gaussian channel in this high-energy regime:

**Corollary 2.** *For a single-mode Gaussian channel  $\Phi$  with parameters  $(\tau \neq 0, y > 0)$ , there exists a fiducial channel  $\Phi^F$  as defined in Theorem 1 such that*

$$\begin{aligned} C^G(\Phi, \bar{N}) &= C^G(\Phi_{(\tau, y, s)}^F, \bar{N}) \\ &= g\left(|\tau|\bar{N} + y \cosh(2s) + \frac{|\tau|-1}{2}\right) - g\left(y + \frac{|\tau|-1}{2}\right), \\ \text{if } \bar{N} \geq \bar{N}_{\text{thr}} &= \frac{1}{2} \left( e^{2|s|} + \frac{2y}{|\tau|} \sinh(2|s|) - 1 \right). \end{aligned} \quad (14)$$

The proof is presented in [43]. Note that the energy threshold  $\bar{N}_{\text{thr}}$  depends on the parameter  $s$  characterizing the fiducial channel  $\Phi_{(\tau, y, s)}^F$ . For thermal channels  $\Phi^{\text{TH}} = \Phi_{(\tau, y, 0)}^F$ , the threshold  $\bar{N}_{\text{thr}}$  vanishes and the additivity holds on the entire energy range. Then, Eq. (14) coincides with previously derived expressions for particular cases [4, 16]. In Fig. 1, we illustrate an example of the domain where  $\bar{N} \geq \bar{N}_{\text{thr}}$ , hence where Eq. (14) holds. Note, that Eq. (14) becomes the actual capacity  $C(\Phi, \bar{N})$  (for  $\bar{N} \geq \bar{N}_{\text{thr}}$ ) of an arbitrary single-mode Gaussian channel  $\Phi$  provided that the vacuum state is proven to minimize the output entropy of a single use of an ideal amplification channel [27, 38].

*Upper bounds.*—Recently, upper bounds have been derived on the capacity of phase-insensitive channels, i.e.  $\Phi^{\text{TH}}$  with  $\tau \geq 0$  [24, 39]. Using Corollary 2, we can generalize it to any Gaussian channel in this high-energy regime:

**Corollary 3.** *For a single-mode Gaussian channel  $\Phi$  with parameters  $(\tau > 0, y > 0)$  and  $\bar{N} \geq \bar{N}_{\text{thr}}$ ,*

$$\begin{aligned} C^G(\Phi, \bar{N}) &\leq C(\Phi, \bar{N}) \leq \bar{C} \leq C^G(\Phi, \bar{N}) + \frac{1}{\ln 2}, \\ \bar{C} &= g\left(\frac{2\tau\bar{N} + (2y+1-\tau) \sinh^2 s}{2y+1+\tau}\right), \end{aligned} \quad (15)$$

where  $C^G(\Phi, \bar{N})$  is stated in Eq. (14).

*Proof.* The fiducial channel corresponding to  $\Phi$  can by definition be decomposed into  $\Phi_{(\tau, y, s)}^F = \Phi_{(G, \frac{G-1}{2}, s)}^F \circ \Phi_{(T, \frac{1-T}{2}, s)}^F$  with  $T = 2\tau/(2y + \tau + 1)$  [see Fig. 2 and Table I]. Then, we can upper bound the capacity of  $\Phi_{(\tau, y, s)}^F$  by the capacity of the first channel, i.e.

$$C(\Phi, \bar{N}) = C\left(\Phi_{(\tau, y, s)}^F, \bar{N}\right) \leq C\left(\Phi_{(T, \frac{1-T}{2}, s)}^F, \bar{N}\right) \leq \bar{C},$$

where  $\bar{C} = g(T\bar{N} + (1-T) \sinh^2 s)$  [18]. Then we define

$$\Delta(s) \equiv \bar{C} - C^G = g[A(B+1)^{-1}] - g(A+B \cosh^2 s) + g(B),$$

where  $A = \tau\bar{N} + [y - \frac{\tau-1}{2}] \sinh^2 s$  and  $B = y + \frac{\tau-1}{2}$ . It was shown in [24] that  $\Delta(0) < 1/\ln 2$ . Since  $\Delta(s) \leq \Delta(0)$ ,  $\forall s$ , the Corollary is proven.  $\square$

Note that for  $\tau < 0$  we can state a similar upper bound on the capacity,  $C(\Phi, \bar{N}) \leq \bar{C}$ , where  $\bar{C}$  is given by Eq. (15) with the replacement  $y \rightarrow -y$ . However, in this case the last inequality in Eq. (15) does not hold. The upper bounds of [39] can also be extended in a similar fashion as shown in [43].

*Conclusions.*—We have shown that an arbitrary single-mode Gaussian channel is either equivalent to a newly defined fiducial channel preceded by a phase-shift and followed by a Gaussian unitary, or can be obtained in a proper limit of this combination. This is exploited to reduce the energy-constrained classical capacity of any single-mode Gaussian channel to that of the fiducial channel. We give an analytical solution to the Gaussian capacity above some energy threshold, where additivity can be proven, and show that in this case the classical capacity does not exceed it by more than  $1/\ln 2$ .

*Acknowledgements.*—J.S. is grateful to Vittorio Giovannetti for clarifications on his work and acknowledges a financial support from the Belgian FRIA foundation. The authors also acknowledge financial support from the F.R.S.-FNRS under the Eranet project HIPERCOM, from the Interuniversity Attraction Poles program of the Belgian Science Policy Office under grant IAP P7-35 “Photonics@be”, from the Brussels Capital Region under the project CRYPTASC, from the ULB under the programme “Ouvertures internationales”, and from the Humboldt foundation.

- 
- [1] C. M. Caves and P. D. Drummond, *Rev. Mod. Phys.* **66**, 481 (1994).
- [2] X.-B. Wang, T. Hiroshima, A. Tomita, and M. Hayashi, *Phys. Rep.* **448**, 1 (2007).
- [3] V. Giovannetti, S. Guha, S. Lloyd, L. Maccone, J. H. Shapiro, and H. P. Yuen, *Phys. Rev. Lett.* **92**, 027902 (2004).
- [4] A. S. Holevo, M. Sohma, and O. Hirota, *Phys. Rev. A* **59**, 1820 (1999).
- [5] G. Bowen, I. Devetak, and S. Mancini, *Phys. Rev. A* **71**, 034310 (2005).
- [6] V. Giovannetti and S. Mancini, *Phys. Rev. A* **71**, 062304 (2005).
- [7] N. J. Cerf, J. Clavareau, C. Macchiavello, and J. Roland, *Phys. Rev. A* **72**, 042330 (2005).
- [8] O. V. Pilyavets, V. G. Zborovskii, and S. Mancini, *Phys. Rev. A* **77**, 052324 (2008).
- [9] S. Guha, *Phys. Rev. Lett.* **106**, 240502 (2011).
- [10] R. König and G. Smith, *Nature Photon.* **7**, 142 (2013).
- [11] M. Sohma and O. Hirota, *Phys. Rev. A* **65**, 022319 (2002).
- [12] M. Sohma and O. Hirota, *Phys. Rev. A* **68**, 022303 (2003).
- [13] M. Sohma and O. Hirota, *Phys. Rev. A* **76**, 024303 (2007).
- [14] M. Sohma and O. Hirota, e-print arXiv:quant-ph/0105042.
- [15] A. S. Holevo, *Russ. Math. Surveys* **53**, 1295 (1998).
- [16] A. S. Holevo and R. Werner, *Phys. Rev. A* **63**, 1 (2001).
- [17] T. Hiroshima, *Phys. Rev. A* **73**, 012330 (2006).
- [18] C. Lupo, O. V. Pilyavets, and S. Mancini, *New J. Phys.* **11**, 063023 (2009).
- [19] J. Schäfer, D. Daems, E. Karpov, and N. J. Cerf, *Phys. Rev. A* **80**, 062313 (2009).
- [20] O. V. Pilyavets, C. Lupo, and S. Mancini, *IEEE Trans. Inf. Theory* **58**, 6126 (2012).
- [21] J. Schäfer, E. Karpov, and N. J. Cerf, in *Proceedings of SPIE* (Bellingham, WA, 2010), vol. 7727, p. 77270J.
- [22] J. Schäfer, E. Karpov, and N. J. Cerf, *Phys. Rev. A* **84**, 032318 (2011).
- [23] J. Schäfer, E. Karpov, and N. J. Cerf, *Phys. Rev. A* **85**, 012322 (2012).
- [24] R. König and G. Smith, *Phys. Rev. Lett.* **110**, 040501 (2013).
- [25] V. Giovannetti, S. Guha, S. Lloyd, L. Maccone, and J. H. Shapiro, *Phys. Rev. A* **70**, 1 (2004).
- [26] M. De Gosson, *Symplectic Geometry and Quantum Mechanics: Operator Theory, Advances and Applications* (Cambridge, MA: Birkhäuser, 2006), vol. 166.
- [27] R. García-Patrón, C. Navarrete-Benlloch, S. Lloyd, J. H. Shapiro, and N. J. Cerf, *Phys. Rev. Lett.* **108**, 110505 (2012).
- [28] A. S. Holevo, *Probl. Inf. Trans.* **44**, 3 (2008).
- [29] F. Caruso, V. Giovannetti, and A. S. Holevo, *New J. Phys.* **8**, 310 (2006).
- [30] A. S. Holevo, *Probl. Inf. Trans.* **43**, 1 (2007).
- [31] J. S. Ivan, K. K. Sabapathy, and R. Simon, *Phys. Rev. A* **84**, 042311 (2011).
- [32] S. L. Braunstein, *Phys. Rev. Lett.* **80**, 4084 (1998).
- [33] J. Fiurasek, N. J. Cerf, and E. S. Polzik, *Phys. Rev. Lett.* **93**, 180501 (2004).
- [34] B. Schumacher and M. D. Westmoreland, *Phys. Rev. A* **56**, 131 (1997).
- [35] M. B. Hastings, *Nature Phys.* **5**, 255 (2009).
- [36] P. W. Shor, *J. Math. Phys.* **43**, 4334 (2002).
- [37] A. S. Holevo and M. E. Shirokov, *Comm. Math. Phys.* **249**, 417 (2004).
- [38] R. García-Patrón and N. J. Cerf, in preparation (2013).
- [39] V. Giovannetti, S. Lloyd, L. Maccone, and J. H. Shapiro, e-print arXiv:1210.3300v1 [quant-ph].
- [40] M. M. Wolf, G. Giedke, O. Krüger, R. F. Werner, and J. I. Cirac, *Phys. Rev. A* **69**, 052320 (2004).
- [41] S. L. Braunstein, *Phys. Rev. A* **71**, 055801 (2005).
- [42] In the rest of the paper, we refer to the classical capacity simply as the *capacity*. Similarly, we use the term *Gaussian capacity* to denote the Gaussian classical capacity. The latter corresponds to the classical capacity obtained with a restriction to Gaussian individual symbol states and Gaussian averaged states.
- [43] See supplemental material for more details on Table I as well as on the derivation of Theorem 1 and its corollaries.

## Supplemental Material

### Physical representation of channels $\Phi^{\text{CS}}$ and $\Phi^{\text{SQ}}$

In the following we explain the physical schemes of the channels  $\Phi^{\text{CS}}$  and  $\Phi^{\text{SQ}}$  as depicted in Fig. 2 (b) and (c). First, we discuss the main “building block” of these schemes, namely the CV-CNOT gate acting on a two-mode state with CM  $V_{\text{in,th}} \equiv V_{\text{in}} \oplus V_{\text{th}}$ , consisting of a general input mode with CM  $V_{\text{in}}$  and an ancillary mode being in a thermal state with CM  $V_{\text{th}}$ , i.e.

$$V_{\text{in}} = \begin{pmatrix} v_q & v_{qp} \\ v_{qp} & v_p \end{pmatrix}, \quad V_{\text{th}} = \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix}. \quad (16)$$

The ancillary mode with CM  $V_{\text{th}}$  in case of the channel  $\Phi^{\text{CS}}$  is in fact the output of the two-mode squeezer with gain  $G$  [Fig. 2 (b)], after having traced over the second ancillary mode, therefore,  $y = G - 1/2$ . The action of the symplectic transformation

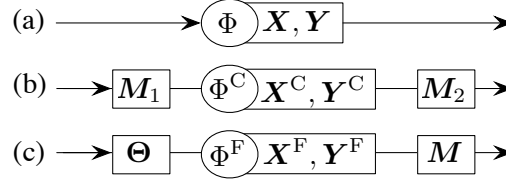


FIG. 3. Equivalence of (a) an arbitrary Gaussian channel  $\Phi$ , (b) the canonical decomposition containing a canonical channel  $\Phi^C$  and (c) the decomposition in terms of the fiducial channel  $\Phi^F$  as stated in Theorem 1.

$S_{\text{CNOT}}$  corresponding to the CV-CNOT on the joint state reads [32, 33]

$$S_{\text{CNOT}} V_{\text{in,th}} S_{\text{CNOT}}^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_q & v_{qp} & 0 & 0 \\ v_{qp} & v_p & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & y \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} v_q & v_{qp} & v_q & 0 \\ v_{qp} & v_p + y & v_{qp} & -y \\ v_q & v_{qp} & y + v_q & 0 \\ 0 & -y & 0 & y \end{pmatrix}. \quad (17)$$

Tracing over the first mode [as shown in Fig. 2 (b)] leads to the output for the classical signal channel  $\Phi^{\text{CS}}$ . Tracing over the second mode [as shown in Fig. 2 (c)] and taking the ancillary mode to be in a pure state ( $y = 1/2$ ) leads to the output of the single quadrature additive noise channel  $\Phi^{\text{SQ}}$ .

#### Proof of Theorem 1 and method how to obtain new decomposition

**Theorem 1.** *For a single-mode Gaussian channel  $\Phi$  defined by matrices  $\mathbf{X}$  and  $\mathbf{Y}$  with  $\tau \neq 0$  and  $y > 0$ , there exists a fiducial channel  $\Phi^F$  defined by matrices  $\mathbf{X}_F(\tau)$ ,  $\mathbf{Y}_F(y, s)$  with  $\tau$  and  $y$  obtained from Eq. (3), a symplectic transformation  $\mathbf{M}$ , and a rotation in phase space  $\Theta$  such that*

$$\mathbf{X} = \mathbf{M} \mathbf{X}_F(\tau) \Theta, \quad \mathbf{Y} = \mathbf{M} \mathbf{Y}_F(y, s) \mathbf{M}^T, \quad (18)$$

where the explicit dependencies of  $\mathbf{M}$ ,  $\Theta$ , and  $s$  on the parameters of the channel  $\Phi$  are presented in Eqs.(26)-(36).

*Proof.* The action of the channel  $\Phi$  on an input CM  $\mathbf{V}$  reads as in Eq. (1)

$$\Phi(\mathbf{V}) = \mathbf{X} \mathbf{V} \mathbf{X}^T + \mathbf{Y}, \quad (19)$$

where  $\mathbf{X}$  is a real  $2 \times 2$  matrix and  $\mathbf{Y}$  a real, symmetric and non-negative  $2 \times 2$  matrix. In the main text we defined the parameters

$$\tau = \det \mathbf{X}, \quad y = \sqrt{\det \mathbf{Y}}, \quad (20)$$

which have to satisfy

$$y \geq \frac{|\tau - 1|}{2}, \quad (21)$$

in order for the map to be physical. In [29–31] it was stated that for any Gaussian channel  $\Phi$  there exists a canonical decomposition  $U_2 \circ \Phi^C \circ U_1$ , where  $\Phi^C$  is a map belonging to one of the seven canonical types that are stated in the main text in Table I. The corresponding action on the CM reads

$$\Phi(\mathbf{V}) = \mathbf{M}_2 (\mathbf{X}_C \mathbf{M}_1 \mathbf{V} \mathbf{M}_1^T \mathbf{X}_C + \mathbf{Y}_C) \mathbf{M}_2^T, \quad (22)$$

where  $\mathbf{X}_C, \mathbf{Y}_C$  are the matrices defining the canonical channels (see Table I in the main text). In the following we obtain the new decomposition in terms of the fiducial channel as stated in the Theorem and furthermore, confirm Eq. (22). The proof is structured as follows. For given matrices  $\mathbf{X}, \mathbf{Y}$  we have to distinguish three cases which depend on the ranks of  $\mathbf{X}$  and  $\mathbf{Y}$  and correspond to canonical decompositions for which  $\Phi^C$  is either  $\Phi^{\text{TH}}$ ,  $\Phi^{\text{SQ}}$  or  $\Phi^{\text{CS}}$ . In the first case our new decomposition will contain finite squeezing operations, while for the other two cases the new decomposition is shown to be valid in a proper limit of infinite squeezing.

For the following calculations we define the symplectic matrices corresponding to a rotation and a squeezing operations, i.e.

$$\mathbf{O}(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad \mathbf{S}(s) = \begin{pmatrix} e^s & 0 \\ 0 & e^{-s} \end{pmatrix}, \quad (23)$$

where we sometimes omit the explicit dependence on the rotation angle or squeezing parameter. For the given CM  $\mathbf{Y}$  one can find a rotation  $\mathbf{\Theta}_Y = \mathbf{O}(\theta_Y)$ , such that  $\mathbf{\Theta}_Y^\top \mathbf{Y} \mathbf{\Theta}_Y = \text{diag}(y_1, y_2)$ , where  $y_1, y_2 \geq 0$  are the eigenvalues of  $\mathbf{Y}$ . Since matrix  $\mathbf{X}$  is always real it has a singular value decomposition (SVD)

$$\mathbf{X} = \mathbf{\Theta}_{1X} \Lambda_X \mathbf{J} \mathbf{\Theta}_{2X}, \quad (24)$$

where  $\mathbf{\Theta}_{1X} = \mathbf{O}(\theta_{1X})$  and  $\mathbf{\Theta}_{2X} = \mathbf{O}(\theta_{2X})$  are rotation matrices. Here

$$\Lambda_X = \text{diag}(x_1, x_2), \quad \mathbf{J} = \begin{cases} \mathbb{I} & \text{if } \tau \geq 0 \\ \sigma_z & \text{if } \tau < 0 \end{cases}, \quad (25)$$

where  $x_1, x_2 \geq 0$  are the singular values and  $\sigma_z = \text{diag}(1, -1)$ . Using equality  $\det \mathbf{X} = \det(\Lambda_X \mathbf{J})$  and Eq. (20) we get  $\tau = \pm x_1 x_2$  and  $y = \sqrt{y_1 y_2}$ . The condition on the determinants of  $\mathbf{X}$  and  $\mathbf{Y}$  stated in Eq. (21) allows us to exclude the following combinations of ranks because they are non-physical:  $(\text{rank}(\mathbf{X}), \text{rank}(\mathbf{Y})) \notin \{(0, 0), (0, 1), (1, 0), (1, 1)\}$ . The physically allowed combinations of ranks therefore read  $(\text{rank}(\mathbf{X}), \text{rank}(\mathbf{Y})) \in \{(0, 2), (2, 0), (2, 2), (1, 2), (2, 1)\}$ . Below we treat the first three ‘‘physical’’ couples together and the other two individually.

**1)** First, we derive the relations for the case  $\text{rank}(\mathbf{X}) = \text{rank}(\mathbf{Y}) = 2$ . The latter implies that  $x_1, x_2, y_1, y_2 \neq 0$ . Then we can construct the squeezing operation  $\mathbf{S}_Y = \mathbf{S}(s_Y)$ , with  $s_Y = \frac{1}{4} \ln(y_1/y_2)$  such that  $\mathbf{S}_Y^{-1} \text{diag}(y_1, y_2) \mathbf{S}_Y^{-1} = \text{diag}(y, y)$ . This implies that

$$\mathbf{Y} = \mathbf{\Theta}_Y \mathbf{S}_Y \mathbf{Y}_{\text{TH}} \mathbf{S}_Y \mathbf{\Theta}_Y^\top = y \mathbf{\Theta}_Y \mathbf{S}_Y^2 \mathbf{\Theta}_Y^\top, \quad \mathbf{Y}_{\text{TH}} = \text{diag}(y, y). \quad (26)$$

Here the symplectic transformation  $\mathbf{\Theta}_Y \mathbf{S}_Y$  realizes the symplectic diagonalization of  $\mathbf{Y}$ , where  $y$  is the symplectic eigenvalue. Furthermore, we can define a squeezing operation  $\mathbf{S}_X = \mathbf{S}(s_X)$ , with  $s_X = \frac{1}{2} \ln(x_1/x_2)$ , such that Eq. (24) can be written as

$$\mathbf{X} = \mathbf{\Theta}_{1X} \mathbf{S}_X \mathbf{X}_{\text{TH}} \mathbf{\Theta}_{2X}, \quad \mathbf{X}_{\text{TH}} = \begin{pmatrix} \sqrt{|\tau|} & 0 \\ 0 & \text{sgn}(\tau) \sqrt{|\tau|} \end{pmatrix}. \quad (27)$$

Notice that the matrix  $\mathbf{X}_{\text{TH}}$  has the property that

$$\mathbf{X}_{\text{TH}} \mathbf{O}(\theta) = \mathbf{O}(\text{sgn}(\tau)\theta) \mathbf{X}_{\text{TH}}. \quad (28)$$

Now we multiply  $\mathbf{Y}$  in Eq. (26) from both sides with the identity matrix in the form  $\mathbb{I} = \mathbf{M} \mathbf{\Theta}_F \mathbf{S}_X^{-1} \mathbf{\Theta}_{1X}^\top$ ,

$$\mathbb{I} \mathbf{Y} \mathbb{I} = y \mathbf{M} \mathbf{\Theta}_F \mathbf{S}_X^{-1} \mathbf{\Theta}_{1X}^\top \mathbf{\Theta}_Y \mathbf{S}_Y^2 \mathbf{\Theta}_Y^\top \mathbf{\Theta}_{1X} \mathbf{S}_X^{-1} \mathbf{\Theta}_F^\top \mathbf{M}^\top, \quad (29)$$

where

$$\mathbf{M} = \mathbf{\Theta}_{1X} \mathbf{S}_X \mathbf{\Theta}_F^\top. \quad (30)$$

Here  $\mathbf{\Theta}_F$  is a rotation that must be chosen in way such that matrix  $\mathbf{Y}_F$  is diagonal, i.e.

$$\mathbf{Y}_F = y \mathbf{\Theta}_F \mathbf{S}_X^{-1} \mathbf{\Theta}_{1X}^\top \mathbf{\Theta}_Y \mathbf{S}_Y^2 \mathbf{\Theta}_Y^\top \mathbf{\Theta}_{1X} \mathbf{S}_X^{-1} \mathbf{\Theta}_F^\top = y \text{diag}(e^{2s}, e^{-2s}), \quad (31)$$

which leads to  $\mathbf{Y} = \mathbf{M} \mathbf{Y}_F \mathbf{M}^\top$ . The explicit expression for the squeezing parameter  $s$  reads

$$\begin{aligned} s &= \frac{1}{2} \ln \left[ \frac{1}{4} e^{-2(s_X - s_Y)} (\tilde{\xi} + \sqrt{-16e^{4(s_X - s_Y)} + \xi^2}) \right], \\ \xi &= (1 + e^{-4s_Y})(1 + e^{4s_X}) + (-1 + e^{-4s_Y})(-1 + e^{4s_X}) \cos(2(\theta_Y - \theta_{1X})), \\ \tilde{\xi} &= (1 + e^{-4s_Y})(-1 + e^{4s_X}) + (-1 + e^{-4s_Y})(1 + e^{4s_X}) \cos(2(\theta_Y - \theta_{1X})). \end{aligned} \quad (32)$$

The angle  $\theta_F$  of rotation  $\mathbf{\Theta}_F = \mathbf{O}(\theta_F)$  reads

$$\theta_F = \arcsin \left( \frac{\text{sgn}(\lambda)}{\sqrt{1 + \lambda^2}} \right), \quad (33)$$

where

$$\lambda = \frac{e^{-2s_X} (\tilde{\xi} - \sqrt{-16e^{4(s_X-s_Y)} + \xi^2})}{2 \sin(2(\theta_Y - \theta_{1X}))(-1 + e^{-4s_Y})}. \quad (34)$$

Using definition  $\mathbf{X}_F = \mathbf{X}_{TH}$  [see Eq. (7) in the main text] and Eq. (28) one can rewrite Eq. (27) as

$$\mathbf{X} = \Theta_{1X} \mathbf{S}_X \Theta_F^\top \Theta_F \mathbf{X}_F \Theta_{2X} = \Theta_{1X} \mathbf{S}_X \Theta_F^\top \mathbf{X}_F \mathbf{O}(\text{sgn}(\tau)\theta_F) \Theta_{2X} = \mathbf{M} \mathbf{X}_F \Theta, \quad (35)$$

where

$$\Theta = \mathbf{O}(\text{sgn}(\tau)\theta_F + \theta_{2X}). \quad (36)$$

In summary, we found that

$$\mathbf{X} = \mathbf{M} \mathbf{X}_F \Theta, \quad \mathbf{Y} = \mathbf{M} \mathbf{Y}_F \mathbf{M}^\top, \quad (37)$$

and we have proven the theorem.

Now we extend the theorem to other combinations of ranks of  $\mathbf{X}$  and  $\mathbf{Y}$ . Let us consider the case  $\text{rank}(\mathbf{X}) = 2, \text{rank}(\mathbf{Y}) = 0$ . Since  $\mathbf{Y} = 0$  it follows that  $y = 0$ , which together with Eq. (21) implies that  $\tau = 1$ . Therefore, the channel is unitarily equivalent to the perfect transmission channel. All relations derived above are found in the same way where one has to fix  $s_Y = \theta_Y = 0$ , which leads to  $\mathbf{S}_Y = \Theta_Y = \mathbb{I}$ .

The third case corresponds to  $\text{rank}(\mathbf{X}) = 0, \text{rank}(\mathbf{Y}) = 2$ . It can also be treated using the above relations. In this case  $\mathbf{X} = 0, \tau = 0$  which together with Eq. (21) implies that  $y \geq 1/2$ . This channel is unitarily equivalent to the zero-transmission channel and has trivially a capacity equal to zero. The decomposition containing the fiducial channel is found above where one has to fix  $s_X = \theta_{1X} = \theta_{2X} = 0$ .

We remark that by inserting Eqs. (26) and (27) in Eq. (19) one obtains the canonical decomposition  $\Phi = U_2 \circ \Phi^{\text{TH}} \circ U_1$ , which in terms of the symplectic transformations reads as in Eq. (22), with

$$\mathbf{X}_C = \mathbf{X}_{TH}, \quad \mathbf{Y}_C = \mathbf{Y}_{TH}, \quad \mathbf{M}_1 = \mathbf{S}_Y^{-1} \Theta_Y'^\top \Theta_{1X}' \mathbf{S}_X \Theta_{2X}, \quad \mathbf{M}_2 = \Theta_Y \mathbf{S}_Y, \quad (38)$$

where  $\Theta_Y' = \mathbf{O}(\text{sgn}(\tau)\theta_Y)$  and  $\Theta_{1X}' = \mathbf{O}(\text{sgn}(\tau)\theta_{1X})$ . This confirms that for  $(\text{rank}(\mathbf{X}), \text{rank}(\mathbf{Y})) \in \{(2, 2), (0, 2), (2, 0)\}$  the physical action of  $\Phi$  indeed corresponds (up to unitaries) to the action of  $\Phi^{\text{TH}}$ . In Fig. 3 we sketched the equivalences found above.

**2)** Next, we consider the case  $\text{rank}(\mathbf{X}) = 2, \text{rank}(\mathbf{Y}) = 1$ . This implies  $y = 0$  and together with Eq. (21) that  $\tau = 1$ . The eigenvalues of  $\mathbf{Y}$  now read  $y_1 = 0, y_2 > 0$  (the case  $y_1 > 0, y_2 = 0$  follows the same treatment). Similarly to the case  $\text{rank}(\mathbf{Y}) = 2$  one can find a rotation  $\Theta_Y$  such that  $\Theta_Y^\top \mathbf{Y} \Theta_Y = \text{diag}(0, y_2)$ . Then, one can construct a squeezing operation  $\Theta_Y$  with  $s_Y = -\frac{1}{2} \ln(2y_2)$  which yields

$$\mathbf{Y} = \Theta_Y \mathbf{S}_Y \mathbf{Y}_{SQ} \mathbf{S}_Y \Theta_Y^\top, \quad \mathbf{Y}_{SQ} = \text{diag}\left(0, \frac{1}{2}\right), \quad (39)$$

The matrix  $\mathbf{Y}_{SQ}$  can be recovered with an additional squeezer  $\mathbf{S}_T = \mathbf{S}(s_T)$  in the limit of infinite squeezing, i.e.  $\mathbf{Y}_{SQ} = \lim_{s_T \rightarrow \infty} \frac{1}{2} e^{-2s_T} \mathbf{S}_T^{-2}$  from which follows

$$\mathbf{Y} = \lim_{s_T \rightarrow \infty} \frac{1}{2} e^{-2s_T} \Theta_Y \mathbf{S}_Y \mathbf{S}_T^{-2} \mathbf{S}_Y \Theta_Y^\top = \lim_{s_T \rightarrow \infty} \frac{1}{2} e^{-2s_T} \Theta_Y \mathbf{S}_{YT}^2 \Theta_Y^\top, \quad (40)$$

where  $\mathbf{S}_{YT} = \mathbf{S}(s_Y - s_T)$ . Since  $\text{rank}(\mathbf{X}) = 2$  we can decompose  $\mathbf{X}$  as in Eq. (27) but with the simplification  $\tau = 1$ , i.e.

$$\mathbf{X} = \Theta_{1X} \mathbf{S}_X \mathbf{X}_{SQ} \Theta_{2X}, \quad \mathbf{X}_{SQ} = \mathbb{I}, \quad (41)$$

We observe that we can replace  $\mathbf{X}_{SQ} = \mathbf{X}_F$ , where  $\mathbf{X}_F$  is as defined as above with  $\tau = 1$ . Thus, we get the same decomposition as stated in Eq. (27). Now one can recover both matrices  $\mathbf{X}, \mathbf{Y}$  as a limiting case of Eq. (37), namely,

$$\mathbf{X} = \lim_{s_T \rightarrow \infty} \mathbf{M} \mathbf{X}_F \Theta, \quad \mathbf{Y} = \lim_{s_T \rightarrow \infty} \mathbf{M} \mathbf{Y}_F \mathbf{M}^\top, \quad (42)$$

where in the definitions of  $\mathbf{M}$  (30),  $\Theta$  (36) and  $\mathbf{Y}_F$  (31) in one has to make replacements  $\tau \rightarrow 1, s_Y \rightarrow s_Y - s_T$  and  $y \rightarrow \frac{1}{2} e^{-2s_T}$ . We remark that this replacement only effects matrix  $\mathbf{Y}_F$  and rotations  $\Theta_F$  and  $\Theta$ . Thus, we found both matrices  $\mathbf{X}$  and  $\mathbf{Y}$  as a limiting as of the decomposition stated in the theorem.



Note that by inserting Eqs. (41) and (39) into Eq. (19) we find the canonical decomposition  $\Phi = U_2 \circ \Phi^{\text{SQ}} \circ U_1$ , which in terms of the symplectic transformations is given by Eq. (22), with

$$\mathbf{X}_C = \mathbf{X}_{\text{SQ}}, \quad \mathbf{Y}_C = \mathbf{Y}_{\text{SQ}}, \quad \mathbf{M}_1 = \mathbf{S}_Y^{-1} \mathbf{\Theta}_Y^\top \mathbf{\Theta}_{1X} \mathbf{S}_X \mathbf{\Theta}_{2X}, \quad \mathbf{M}_2 = \mathbf{\Theta}_Y \mathbf{S}_Y, \quad (43)$$

showing that the physical action of  $\Phi$  in this case corresponds (up to unitaries) to the action of  $\Phi^{\text{SQ}}$ .

3) The last case to consider is  $\text{rank}(\mathbf{X}) = 1, \text{rank}(\mathbf{Y}) = 2$ . Since in this case  $\tau = 0$  it follows from Eq. (21) that  $y \geq \frac{1}{2}$ . The SVD of  $\mathbf{X}$  now reads  $\mathbf{X} = \mathbf{\Theta}_{1X} \text{diag}(x_1, 0)$  (the case  $x_1 = 0, x_2 > 0$  follows the same treatment). One can define  $\mathbf{S}_X = \mathbf{S}(s_X)$  with  $s_X = \ln(x_1)$  such that

$$\mathbf{X} = \mathbf{\Theta}_{1X} \mathbf{S}_X \mathbf{X}_{\text{CS}}, \quad \mathbf{X}_{\text{CS}} = \text{diag}(1, 0). \quad (44)$$

Since  $\mathbf{X}_{\text{CS}}$  can be expressed as  $\mathbf{X}_{\text{CS}} = \lim_{s_T \rightarrow \infty} e^{-s_T} \mathbf{S}_T$ , where  $\mathbf{S}_T = \mathbf{S}(s_T)$ , Eq. (44) becomes

$$\mathbf{X} = \lim_{s_T \rightarrow \infty} e^{-s_T} \mathbf{\Theta}_{1X} \mathbf{S}_X \mathbf{S}_T = \lim_{s_T \rightarrow \infty} e^{-s_T} \mathbf{\Theta}_{1X} \mathbf{S}_{XT}, \quad (45)$$

where  $\mathbf{S}_{XT} = \mathbf{S}(s_X + s_T)$ . Since  $y \geq \frac{1}{2}$  we find as in the case  $y > 0$  treated above [see derivation of Eq. (26)], a rotation  $\mathbf{\Theta}_Y$  and squeezing  $\mathbf{S}_Y$  such that

$$\mathbf{Y} = \mathbf{\Theta}_Y \mathbf{S}_Y \mathbf{Y}_{\text{CS}} \mathbf{S}_Y^\top \mathbf{\Theta}_Y^\top, \quad \mathbf{Y}_{\text{CS}} = \text{diag}(y, y), \quad y \geq \frac{1}{2}. \quad (46)$$

We observe, that we can recover matrices  $\mathbf{X}, \mathbf{Y}$  as a limiting case of Eq. (37), i.e.

$$\mathbf{X} = \lim_{s_T \rightarrow \infty} \mathbf{M} \mathbf{X}_F \mathbf{\Theta}, \quad \mathbf{Y} = \lim_{s_T \rightarrow \infty} \mathbf{M} \mathbf{Y}_F \mathbf{M}^\top, \quad (47)$$

where in the definitions of  $\mathbf{M}$  (30) and  $\mathbf{\Theta}$  (36) one has to make replacements  $\theta_{2X} \rightarrow 0, s_X \rightarrow s_X + s_T$  and  $\tau \rightarrow e^{-2s_T}$ . Note that this replacement effects  $\mathbf{M}$  but does not effect matrix  $\mathbf{Y}$  stated in Eq. (46). Therefore, we found also for the last case both matrices  $\mathbf{X}, \mathbf{Y}$  as limiting cases of the decomposition stated in the theorem.

The canonical decomposition in this case is found as follows. By inserting Eqs. (46) and (44) in Eq. (19), we obtain

$$\Phi(\mathbf{V}) = \tilde{\mathbf{M}}(\tilde{\mathbf{X}} \mathbf{V} \tilde{\mathbf{X}}^\top + \mathbf{Y}_{\text{CS}}) \tilde{\mathbf{M}}^\top, \quad \tilde{\mathbf{X}} = \mathbf{S}_Y^{-1} \mathbf{\Theta}_Y^\top \mathbf{\Theta}_{1X} \mathbf{S}_X \mathbf{X}_{\text{CS}}, \quad \tilde{\mathbf{M}} = \mathbf{\Theta}_Y \mathbf{S}_Y. \quad (48)$$

For the real  $2 \times 2$  matrix  $\tilde{\mathbf{X}}$  one can again obtain the SVD which leads to  $\tilde{\mathbf{X}} = \tilde{\mathbf{\Theta}}_X \tilde{\mathbf{S}}_X \mathbf{X}_{\text{CS}}$ . Since  $\tilde{\mathbf{S}}_X \mathbf{X}_{\text{CS}} = \mathbf{X}_{\text{CS}} \tilde{\mathbf{S}}_X$  we obtain the canonical decomposition  $\Phi = U_2 \circ \Phi^{\text{CS}} \circ U_1$  in terms of the symplectic transformations as stated in Eq. (22), with

$$\mathbf{X}_C = \mathbf{X}_{\text{CS}}, \quad \mathbf{Y}_C = \mathbf{Y}_{\text{CS}}, \quad \mathbf{M}_1 = \tilde{\mathbf{S}}_X, \quad \mathbf{M}_2 = \mathbf{\Theta}_Y \mathbf{S}_Y \tilde{\mathbf{\Theta}}_X, \quad (49)$$

demonstrating that (up to unitaries) the physical action of  $\Phi$  in this case corresponds to the action of  $\Phi^{\text{CS}}$ . Thus, we extended the theorem to lower rank cases of  $\mathbf{X}$  and  $\mathbf{Y}$ .  $\square$

### Derivation of simplified expression for the one-shot Gaussian capacity

In the following we show that the one-shot Gaussian capacity of a single-mode Gaussian channel  $\Phi$  can be expressed as

$$C_\chi^{\text{G}}(\Phi, \bar{N}) = \max_{\mathbf{V}_{\text{in}}, \mathbf{V}_{\text{mod}}} \{ \chi^{\text{G}}(\bar{\nu}, \nu) \mid \text{Tr}[\mathbf{V}_{\text{in}} + \mathbf{V}_{\text{mod}}] \leq 2\bar{N} + 1 \}, \quad (50)$$

$$\chi^{\text{G}} = g\left(\bar{\nu} - \frac{1}{2}\right) - g\left(\nu - \frac{1}{2}\right), \quad (51)$$

$$g(x) = (x + 1) \log_2(x + 1) - x \log_2 x, \quad (52)$$

where  $g(0) = 0$ ,  $\mathbf{V}_{\text{in}}$  is the CM of a pure Gaussian input state fulfilling  $\det 2\mathbf{V}_{\text{in}} = 1$ ,  $\mathbf{V}_{\text{mod}}$  is the CM of a classical Gaussian distribution used to displace the input state and to generate the modulated input state with CM  $\bar{\mathbf{V}}_{\text{in}} = \mathbf{V}_{\text{in}} + \mathbf{V}_{\text{mod}}$  where  $\text{Tr}[\bar{\mathbf{V}}_{\text{in}}] \leq 2\bar{N} + 1$ . Furthermore,  $\nu = \sqrt{\det \mathbf{V}_{\text{out}}}$  and  $\bar{\nu} = \sqrt{\det \bar{\mathbf{V}}_{\text{out}}}$  are the symplectic eigenvalues of the output and modulated output state with CM  $\mathbf{V}_{\text{out}} = \Phi(\mathbf{V}_{\text{in}})$  and  $\bar{\mathbf{V}}_{\text{out}} = \Phi(\bar{\mathbf{V}}_{\text{in}})$ , respectively. Equation (50) states that among all possible Gaussian sources, where a source is characterized by a measure  $\mu^{\text{G}}(d\alpha, d\mathbf{V}_p)$  over the set of pure Gaussian states  $\hat{\rho}^{\text{G}}(\alpha, \mathbf{V}_p)$  of mean  $\alpha$  and CM  $\mathbf{V}_p$  fulfilling  $\det(2\mathbf{V}_p) = 1$ , the source optimizing the Gaussian capacity of a given Gaussian channel

corresponds to using a single pure Gaussian state  $\hat{\rho}^G(0, \mathbf{V}_{\text{in}})$  with covariance  $\mathbf{V}_{\text{in}}$  fulfilling  $\det(2\mathbf{V}_{\text{in}}) = 1$ , modulated following a Gaussian bivariate distribution with CM  $\mathbf{V}_{\text{mod}}$ .

To achieve our goal we use the fact that the maximization inside the Gaussian capacity definition

$$C_\chi^G(\Phi, \bar{N}) = \max_{\mu^G: \hat{\rho}^G \in \mathcal{E}_N^G} \left[ S(\Phi[\hat{\rho}^G]) - \int \mu^G(d\alpha, d\mathbf{V}_p) S(\Phi[\hat{\rho}^G(\alpha, \mathbf{V}_p)]) \right], \quad (53)$$

can be divided into two different steps. In the first step, among all the sources  $\mu^G(d\alpha, d\mathbf{V}_p)$  belonging to the set  $\mathcal{F}_{\hat{\rho}^G}^G$  sharing the same average output state

$$\hat{\rho}^G = \int \mu^G(d\alpha, d\mathbf{V}_p) \hat{\rho}^G(\alpha, \mathbf{V}_p), \quad (54)$$

we maximize the modified Holevo quantity

$$\tilde{\chi}(\Phi, \bar{N}, \hat{\rho}^G) = S(\Phi[\hat{\rho}^G]) - \min_{\mu^G \in \mathcal{F}_{\hat{\rho}^G}^G} \int \mu^G(d\alpha, d\mathbf{V}_p) S(\Phi[\hat{\rho}^G(\alpha, \mathbf{V}_p)]). \quad (55)$$

In the second and final step we optimize  $\tilde{\chi}(\Phi, \bar{N}, \hat{\rho}^G)$  over the average input state  $\hat{\rho}^G$  satisfying the energy constraint  $\bar{N}$ , thus obtaining  $C_\chi^G(\Phi, \bar{N})$ . It is well known that the minimum of the average output entropy appearing in equation (55) can be rewritten as the Gaussian entanglement of formation  $E^G[\sigma_{\text{BE}}]$  of a given bipartite mixed state  $\sigma_{\text{BE}} = U_\Phi \hat{\rho}_B^G \otimes |0\rangle\langle 0|_E U_\Phi^\dagger$  resulting from sending the average input mixed state  $\hat{\rho}^G$  through the unitary dilation  $U_\Phi$  of channel  $\Phi$ , i.e.

$$\min_{\mu^G \in \mathcal{F}_{\hat{\rho}^G}^G} \int \mu^G(d\alpha, d\mathbf{V}_p) S(\Phi[\hat{\rho}^G(\alpha, \mathbf{V}_p)]) = E^G[\sigma_{\text{BE}}], \quad (56)$$

where the equality  $\text{Tr}_E[\sigma_{\text{BE}}] = \Phi[\hat{\rho}^G]$  is guaranteed by the Stinespring dilation of the channel. Indeed, the Gaussian entanglement of formation is defined as

$$E^G[\sigma_{\text{BE}}] = E^G[U_\Phi \hat{\rho}_B^G \otimes |0\rangle\langle 0|_E U_\Phi^\dagger] = \min_{\mu^G \in \mathcal{F}_{\hat{\rho}^G}^G} \left[ \int \mu^G(d\alpha, d\mathbf{V}_p) E[U_\Phi \hat{\rho}_B^G(\alpha, \mathbf{V}_p) \otimes |0\rangle\langle 0|_E U_\Phi^\dagger] \right]. \quad (57)$$

Here  $E[\hat{\rho}^G(\alpha, \mathbf{V}_p)]$  corresponds to the entanglement of a pure bipartite Gaussian state, which is uniquely quantified by the von Neumann entropy of any of its two reduced density operators. This connection, not only simplifies the capacity definition to

$$C_\chi^G(\Phi, \bar{N}) = \max_{\hat{\rho}^G \in \mathcal{E}_N^G} \left[ S(\Phi[\hat{\rho}^G]) - E^G[U_\Phi \hat{\rho}_B^G \otimes |0\rangle\langle 0|_E U_\Phi^\dagger] \right], \quad (58)$$

but also leads to the proof of Eq. (50). In [40] it was shown that

$$E^G[\hat{\rho}^G(\alpha, \mathbf{V})] = \min_{\mathbf{V}_p} \{ E(\hat{\rho}^G(0, \mathbf{V}_p)) | \mathbf{V}_p \leq \mathbf{V} \}, \quad (59)$$

where the infimum is taken over pure Gaussian states of CM  $\mathbf{V}_p$ . This implies the existence of a covariance matrix  $\mathbf{V}_p$  such that the output entropy of pure Gaussian states  $\hat{\rho}^G(\alpha, \mathbf{V}_p)$  achieves the minimum in equation (56) for any  $\alpha$ . Therefore, we chose the state  $\hat{\rho}^G(\alpha, \mathbf{V}_p)$  as the symbol states, i.e.  $\mathbf{V}_{\text{in}} = \mathbf{V}_p$ . Then modulating  $\hat{\rho}^G(\alpha, \mathbf{V}_{\text{in}})$  according to a Gaussian distribution with covariance matrix  $\mathbf{V}_{\text{mod}} = \bar{\mathbf{V}}_{\text{in}} - \mathbf{V}_{\text{in}}$  generates a source with average input state  $\hat{\rho}^G$  saturating the bound of equation (55). Thus, the one-shot Gaussian capacity for a general  $n$ -mode Gaussian channel  $\Phi$  simplifies to

$$C_\chi^G(\Phi, \bar{N}) = \max_{\mathbf{V}_{\text{in}}, \mathbf{V}_{\text{mod}}} \{ S(\Phi(\mathbf{V}_{\text{in}} + \mathbf{V}_{\text{mod}})) - S(\Phi(\mathbf{V}_{\text{in}})) \mid \text{Tr}[\mathbf{V}_{\text{in}} + \mathbf{V}_{\text{mod}}] \leq 2n\bar{N} + 1 \}. \quad (60)$$

The entropy of an  $n$ -mode Gaussian state  $\hat{\rho}^G(\alpha, \mathbf{V})$  can be calculated in terms of the  $n$  symplectic eigenvalues  $\nu_i$  of  $\mathbf{V}$ , namely

$$S(\hat{\rho}^G(\alpha, \mathbf{V})) = \sum_{i=1}^n g\left(\nu_i - \frac{1}{2}\right), \quad (61)$$

where  $g(x)$  is defined in Eq. (50). Thus, the one-shot Gaussian capacity of a general  $n$ -mode Gaussian channel simplifies to

$$C_\chi^G(\Phi, \bar{N}) = \max_{\mathbf{V}_{\text{in}}, \mathbf{V}_{\text{mod}}} \left\{ \left[ \sum_{i=1}^n g\left(\bar{\nu}_i - \frac{1}{2}\right) - g\left(\nu_i - \frac{1}{2}\right) \right] \mid \text{Tr}[\mathbf{V}_{\text{in}} + \mathbf{V}_{\text{mod}}] \leq 2n\bar{N} + 1 \right\}, \quad (62)$$

where  $\nu_i$  and  $\bar{\nu}_i$  are the symplectic eigenvalues of the CM  $\mathbf{V}_{\text{out}} = \Phi(\mathbf{V}_{\text{in}})$  and CM  $\bar{\mathbf{V}}_{\text{out}} = \Phi(\bar{\mathbf{V}}_{\text{in}})$ . For the one mode case Eq. (62) simplifies to Eq. (50).

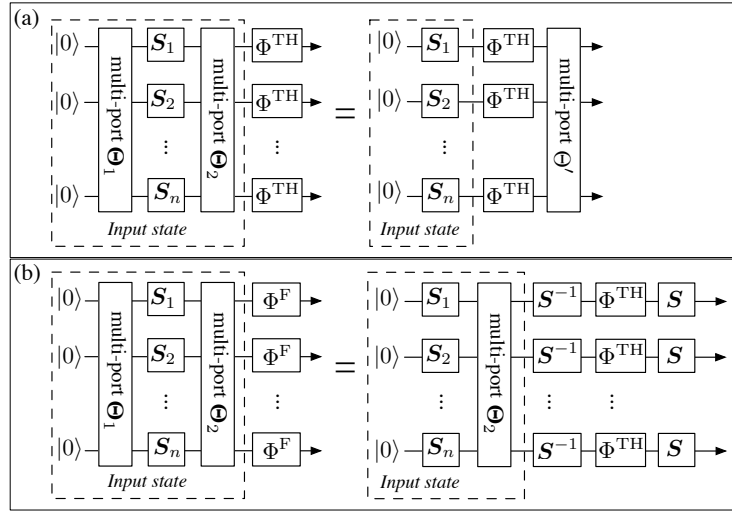


FIG. 4. Using the Bloch Messiah reduction any multimode pure Gaussian state can be generated from the vacuum and a set of single mode squeezers  $S_i$  preceded and followed by linear multi-port interferometers  $\Theta_1, \Theta_2$ . (a) Reduction of the scheme for a collection of channels  $(\Phi^{\text{TH}})^{\otimes n}$  and (b) Reduction for the fiducial channel  $(\Phi^{\text{F}})^{\otimes n}$  (valid above an input energy threshold).

### Proof of Corollary 2

**Corollary 2.** *For a single-mode Gaussian channel  $\Phi$  with parameters  $(\tau \neq 0, y > 0)$ , there exists a fiducial channel  $\Phi^{\text{F}}$  as defined in Theorem 1 such that*

$$C^{\text{G}}(\Phi, \bar{N}) = C^{\text{G}}(\Phi_{(\tau, y, s)}^{\text{F}}, \bar{N}) = g\left(|\tau|\bar{N} + y \cosh(2s) + \frac{|\tau| - 1}{2}\right) - g\left(y + \frac{|\tau| - 1}{2}\right), \quad (63)$$

$$\text{if } \bar{N} \geq \bar{N}_{\text{thr}} = \frac{1}{2} \left( e^{2|s|} + \frac{2y}{|\tau|} \sinh(2|s|) - 1 \right),$$

*Proof.* Corollary 1 can be straightforwardly extended to conclude that  $C^{\text{G}}(\Phi, \bar{N}) = C^{\text{G}}(\Phi^{\text{F}}, \bar{N})$ . This means that we only have to evaluate the Gaussian capacity of the fiducial channel  $\Phi^{\text{F}}$  in order to find the Gaussian capacity for  $\Phi$ . In the following we find  $C^{\text{G}}(\Phi^{\text{F}}, \bar{N})$  explicitly above the input energy threshold  $\bar{N}_{\text{thr}}$ .

The proof is structured as follows. First, we prove that the Gaussian minimum output entropy of thermal channels  $\Phi^{\text{TH}}$  is additive (corresponding to  $\Phi^{\text{F}}$  with  $s = 0$ ). Then we extend this proof to the fiducial channel for input energies above the input energy threshold  $\bar{N}_{\text{thr}}$  (we present a simple and physically motivated proof, which is an alternative to the one in [17]). Then we show that this also implies the additivity of the one-shot Gaussian capacity in this energy domain. Finally, we derive the exact expression for the one-shot Gaussian capacity.

In [41] it was shown that any pure  $n$ -mode Gaussian (input) state can be generated from a set of  $n$  vacuum modes, using  $n$  single-mode squeezers  $S_i$  preceded and followed by a linear multi-port interferometer, corresponding to passive symplectic transformations  $\Theta_1$  and  $\Theta_2$  [see Fig. 4 (a)]. The multi-mode vacuum state with CM  $\mathbb{I}/2$  (where  $\mathbb{I}$  is the  $2n \times 2n$  identity matrix) remains unchanged under the action of the first interferometer  $\Theta_1$  therefore, we can drop it. The action of the channel  $\Phi_{(\tau, y)}^{\text{TH}}$  in terms of symplectic transformation then reads

$$\mathbf{V}_{\text{out}} = \frac{1}{2} \mathbf{X}_{\text{TH}} \Theta_2 \mathbf{S} \mathbb{I} \mathbf{S} \Theta_2^{\text{T}} \mathbf{X}_{\text{TH}} + \mathbf{Y}_{\text{TH}}, \quad (64)$$

where  $\mathbf{S} = \oplus_{i=1}^n S_i$  corresponds to the joint operation of single mode squeezers,  $\Theta_2$  is the symplectic transformation of the second linear multi-port interferometer and

$$\mathbf{X}_{\text{TH}} = \bigoplus_{i=1}^n \begin{pmatrix} \sqrt{|\tau|} & 0 \\ 0 & \text{sgn}(\tau) \sqrt{|\tau|} \end{pmatrix}, \quad \mathbf{Y}_{\text{TH}} = \bigoplus_{i=1}^n y \mathbb{I}. \quad (65)$$

By inserting  $\mathbf{X}_{\text{TH}} \mathbf{X}_{\text{TH}} = \mathbb{I}$  between  $\Theta_2$  and  $\mathbf{S}$  in Eq. (64) we obtain

$$\mathbf{V}_{\text{out}} = \Theta' \left( \frac{1}{2} \mathbf{X}_{\text{TH}} \mathbf{S} \mathbb{I} \mathbf{S} \mathbf{X}_{\text{TH}} + \mathbf{Y}_{\text{TH}} \right) \Theta'^{\text{T}}, \quad (66)$$

where  $\Theta' = X_{\text{TH}}\Theta_2 X_{\text{TH}}$  and  $\Theta' Y_{\text{TH}} \Theta'^T = Y_{\text{TH}}$ . One can confirm easily that  $\Theta' \Omega \Theta'^T = \Omega$  and  $\Theta' \Theta'^T = \mathbb{I}$ , so  $\Theta'$  is just another interferometer at the output. Thus, the general Gaussian input state entering the channel  $(\Phi^{\text{TH}})^{\otimes n}$  is reduced to a product state [left hand side of Fig. 4 (a)].

Now we calculate the entropy of the output state with CM  $V_{\text{out}}$ . Since  $\Theta'$  does not effect its entropy we can drop it. Therefore, we are left with  $n$  vacuum modes entering the set of single-mode squeezers and then individually passing each channel  $\Phi^{\text{TH}}$ . From this follows that the output entropy is additive and thus,

$$\min_{V_{\text{in}}} \frac{1}{n} S(V_{\text{out}}) = \min_{V_{\text{in}}} \frac{1}{n} S\left((\Phi^{\text{TH}})^{\otimes n} \left(\frac{1}{2} S \mathbb{I} S\right)\right) = \min_{V_{\text{in}}} \frac{1}{n} \sum_i S(\Phi^{\text{TH}}(V_{\text{in}i})) = \min_{V_{\text{in}}} \frac{1}{n} \sum_i g\left(\nu_i - \frac{1}{2}\right), \quad (67)$$

where  $V_{\text{in}i} = \frac{1}{2} S_i^2$  is the CM of vacuum mode  $i$  exiting the single mode squeezer  $S_i$  and  $\nu_i = \sqrt{\det(\Phi^{\text{TH}}(V_{\text{in}i}))}$  are the symplectic eigenvalues of the CM of the individual output states. The output entropy is minimized for  $V_{\text{in}i} = \mathbb{I}/2, \forall i$ , i.e. by fixing all one-mode squeezers  $S_i = \mathbb{I}$ . This leads to

$$\min_{V_{\text{in}}} \frac{1}{n} S(V_{\text{out}}) = g\left(y + \frac{|\tau| - 1}{2}\right). \quad (68)$$

Now let us consider the same problem for the fiducial channel  $\Phi^{\text{F}}$  as depicted in Fig. 4 (b). We use again the Bloch-Messiah decomposition to decompose the general multi-mode Gaussian input state. The first interferometer  $\Theta_1$  can again be dropped because it does not effect the  $n$  mode vacuum state. From the definition of the fiducial channel we have the equivalence  $X_{\text{F}} = X_{\text{TH}}, X_{\text{F}} = S X_{\text{TH}} S$ , with  $S = \text{diag}(e^s, e^{-s})$ . This leads to the equality

$$\Phi^{\text{F}}(V) = S(X_{\text{TH}} S^{-1} V S^{-1} X_{\text{TH}} + Y_{\text{TH}}) S. \quad (69)$$

As a consequence we can replace each fiducial channel by a thermal channel preceded by an anti-squeezer and followed by a squeezer [see right hand side of Fig. 4 (b)].

Now we focus again on the minimization of the output entropy. Then, the set of squeezers  $S$  can be dropped since they do not change the entropy and we have no energy constraint on the output. We showed above that the entropy for the joint map  $(\Phi^{\text{TH}})^{\otimes n}$  is minimized by the  $n$ -mode vacuum state. Thus, the multi-mode input state that minimizes the output entropy of the fiducial channel has to be in the  $n$ -mode vacuum state after passing the  $n$  anti-squeezers  $S^{-1}$  [see right hand side of Fig. 4 (b)]. Therefore, one fixes the interferometer  $\Theta_2 = \mathbb{I}$  and choses each squeezer  $S_i$  at the input to undo the anti-squeezers, i.e.  $S_i = S, \forall i$ . Thus, the entropy minimizing  $n$ -mode Gaussian input state for the channel  $(\Phi^{\text{TH}})^{\otimes n}$  is the product state with CM  $V_{\text{in}} = \oplus_{i=1}^n S^2/2$ . It follows that

$$\min_{V_{\text{in}}} \frac{1}{n} S\left((\Phi^{\text{F}})^{\otimes n}(V_{\text{in}})\right) = g\left(y + \frac{|\tau| - 1}{2}\right), \quad (70)$$

where the minimization requires a certain amount of energy to undo each squeezer, which will be taken into account later.

We show now that Eq. (70) leads to the additivity of the one-shot Gaussian capacity of  $\Phi^{\text{F}}$  above the input energy threshold  $\bar{N}_{\text{thr}}$ . Using the expression for the one-shot Gaussian capacity  $C_{\chi}^{\text{G}}(\Phi^{\text{F}}, \bar{N})$  stated in Eq. (60) and using the definition of the Gaussian capacity

$$C^{\text{G}}(\Phi, \bar{N}) = \lim_{n \rightarrow \infty} \frac{1}{n} C_{\chi}^{\text{G}}(\Phi^{\otimes n}, n\bar{N}), \quad (71)$$

we can state the following upper bound:

$$C^{\text{G}}(\Phi^{\text{F}}, \bar{N}) \leq \max_{V_{\text{in}}, V_{\text{mod}}} S(\Phi^{\text{F}}(V_{\text{in}} + V_{\text{mod}})) - \lim_{n \rightarrow \infty} \min_{V_{\text{in}}} \frac{1}{n} S\left((\Phi^{\text{F}})^{\otimes n}(V_{\text{in}})\right), \quad (72)$$

where the first term only needs to be maximized for a single use of the channel due to the subadditivity of the entropy. It is known that a thermal state maximizes the entropy, therefore, the optimal modulated output state is a thermal state carrying the total number of photons, i.e.  $\Phi^{\text{F}}(V_{\text{in}} + V_{\text{mod}}) = \bar{V}_{\text{out}} = \text{diag}(\bar{\nu}, \bar{\nu})$ , where

$$\bar{\nu} = |\tau| \bar{N} + y \cosh(2s) + |\tau|/2. \quad (73)$$

The second term in Eq. (72) was already evaluated above [see Eq. (70)]. In summary, we found

$$\max_{V_{\text{in}}, V_{\text{mod}}} S(\Phi^{\text{F}}(V_{\text{in}} + V_{\text{mod}})) = g\left(|\tau| \bar{N} + y \cosh(2s) + \frac{|\tau| - 1}{2}\right), \quad (74)$$

$$\lim_{n \rightarrow \infty} \min_{V_{\text{in}}} \frac{1}{n} S\left((\Phi^{\text{F}})^{\otimes n}(V_{\text{in}})\right) = g\left(y + \frac{|\tau| - 1}{2}\right). \quad (75)$$

The encoding which realizes both the maximum and the minimum in Eqs. (74) a (75) is given by

$$\mathbf{V}_{\text{in}} = \mathbf{S}^2/2, \quad \mathbf{V}_{\text{mod}} = \bar{\mathbf{V}}_{\text{in}} - \mathbf{V}_{\text{in}}, \quad \bar{\mathbf{V}}_{\text{in}} = \begin{pmatrix} \bar{N} + \frac{1}{2} - \frac{y}{|\tau|} \sinh(2s) & 0 \\ 0 & \bar{N} + \frac{1}{2} + \frac{y}{|\tau|} \sinh(2s) \end{pmatrix}. \quad (76)$$

This encoding can only be realized if

$$\bar{N} \geq \bar{N}_{\text{thr}} = \frac{1}{2} \left( e^{2|s|} + \frac{2y}{|\tau|} \sinh(2|s|) - 1 \right), \quad (77)$$

because for  $\bar{N} < \bar{N}_{\text{thr}}$  it would imply  $\mathbf{V}_{\text{mod}} < 0$  which would be non-physical. Thus, we have shown that

$$C^G(\Phi^F, \bar{N}) = C_\chi^G(\Phi^F, \bar{N}), \quad \bar{N} \geq \bar{N}_{\text{thr}}, \quad (78)$$

where  $C_\chi^G(\Phi^F, \bar{N})$  is the right hand side of Eq. (63). This proves the Corollary.  $\square$

### Additional upper bounds on the classical capacity

The upper bound on the classical capacity stated in Corollary 3 was obtained by generalizing the bounds that were found for thermal channels  $\Phi^{\text{TH}}$  (with  $\tau > 0$ ) in [24]. Recently, additional upper bounds were obtained for the same channels [39] and we extend them now to general channels  $\Phi$  with  $\tau > 0, y > 0$ .

The bounds were obtained by maximizing the first term of the Holevo quantity [defined in Eq. (10) in the main text] and by obtaining a lower bound  $b$  on the second term, i.e.

$$\lim_{n \rightarrow \infty} \frac{1}{n} \min_{\mu} \int \mu(dx) S((\Phi^{\text{TH}})^{\otimes n}[\hat{\rho}_x]) \geq b, \quad (79)$$

where in total six bounds  $b$  are stated in [39]. We stated in the proof of Corollary 2 that the fiducial channel is equivalent to a thermal channel preceded by an anti-squeezer and followed by a squeezer [see Fig. 4 (b)]. The following squeezer does not change the output entropy. Furthermore, one can always undo the preceding squeezer because the bound  $b$  is not subject to an energy constraint. Therefore, any bound  $b$  on the minimal output entropy of the thermal channel is as well an upper bound on the minimal output entropy of the fiducial channel. In addition, the first term of the classical capacity is known to be maximized by a one-mode Gaussian thermal state carrying the total number of photons of the system. This quantity is already given in Eq. (74), i.e.

$$\lim_{n \rightarrow \infty} \frac{1}{n} \max_{\mu: \hat{\rho} \in \mathcal{E}_N} S((\Phi^F)^{\otimes n}[\hat{\rho}]) = g\left(|\tau|\bar{N} + y \cosh(2s) + \frac{|\tau| - 1}{2}\right). \quad (80)$$

Corollary 1 states that  $C(\Phi, \bar{N}) = C(\Phi^F, \bar{N})$ . Therefore, any bound on the classical capacity of the fiducial channel  $\Phi^F$  is also a bound on the classical capacity of an arbitrary channel  $\Phi$ . Thus, we obtain a list of upper bounds on the classical capacity that reads

$$C(\Phi, \bar{N}) \leq g\left(|\tau|\bar{N} + y \cosh(2s) + \frac{|\tau| - 1}{2}\right) - b, \quad \tau > 0, y > 0, \quad (81)$$

where  $b$  has to be taken from [39] and substituted in the latter. Note that with increasing  $s$  those bounds become less and less tight because  $b$  does not depend on  $s$ . However, as in the case of thermal channels [39] some of those bounds in a certain region of channel parameters are tighter than the bound we presented in Corollary 3.